REDUCING THE EQUATIONS OF NONLINEAR NON-STEADY-STATE HIGH-INTENSITY HEAT- AND MASS-TRANSFER TO EQUIVALENT LINEAR EQUATIONS. AN ANALOGY OF THE THEORY OF HIGH-INTENSITY HEAT- AND MASS-TRANSFER

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We have derived the basic forms of the equivalent linear equations for nonlinear high-intensity non-steady-state heat- and mass-transfer. Methods are proposed for the modeling of these processes on the basis of various physical analogies.

The one-dimensional equation of high-intensity non-steady-state processes of heat- and mass-transfer, formulated by A. V. Luikov [1], has the form

$$c\gamma\frac{\partial T}{\partial\tau}+\frac{\lambda}{w_r^2}\frac{\partial^2 T}{\partial\tau^2}=\lambda\frac{\partial^2 T}{\partial x^2}. \quad (1)$$

If we neglect the term with the second derivative, Eq. (1) represents the classical heat-conduction equation in which the well-known paradox is the presence of an infinite rate of heat propagation.

Consideration of the nonlinear temperature properties of the medium, when the functions c, γ, and λ are functions of temperature, leads to the existence of a finite heat-propagation rate [2], which is a function of the material properties and the conditions under which the heat flows are produced.

P. Vernotte [3] indicated the existence of a finite heat-propagation rate for high-intensity non-steady-state processes of heat transfer in rarefied media; A. V. Luikov [4] did the same for the processes of heat- and moisture-transport in capillary-porous media. The rate of heat propagation is given by

$$w_r=\sqrt{\frac{\lambda}{c\gamma\tau_r}}. \quad (2)$$

For metals the velocity constant is on the order of $\tau_r\sim10^{-11}$ sec, while for gases it is on the order of $\tau_r\sim10^{-9}$ sec. For example, for nitrogen the rate of heat-propagation is $w_r\sim150$ m/sec.

It may develop in high-intensity non-steady-state processes of heat- and mass-transfer that the term with the first derivative with respect to time is considerably smaller than the term with the second derivative. In this case, the equation of heat- and mass-transfer is a purely hyperbolic wave equation [1]

$$\frac{\partial^2 T}{\partial\tau^2}=w_r^2\frac{\partial^2 T}{\partial x^2}. \quad (3)$$

In the case of mass transfer, T is understood to refer to the concentration of the medium. If we assume that the heat-propagation rate w_r is finite and constant, Eq. (3) will describe the propagation of the heat- and mass-transfer waves, and the nature of the heat propagation will no longer be diffusive as in the classical theory of heat transfer, but rather it will be wavelike. The methods for the integration of Eq. (3) with the constant w_r are well known, and we will not dwell on these here.

Let us consider the hyperbolic heat-conduction equation (3) in which the heat-propagation rate w_r of (2) is exclusively a function of $\partial T/\partial x$, i.e., of the temperature gradient:

$$w_r=w_r\left(\frac{\partial T}{\partial x}\right). \quad (2a)$$

In this case, Eq. (3) is changed from a linear equation to a quasi-linear heat-conduction equation:

$$\frac{\partial^2 T}{\partial\tau^2}=w_r^2\left(\frac{\partial T}{\partial x}\right)\frac{\partial^2 T}{\partial x^2}. \quad (4)$$

Having introduced the characteristic directions for the temperature field (4), we can bring Eq. (4) to an equivalent system of ordinary differential characteristic equations:

the equations for the first family of characteristics:

$$dx=+w_r d\tau, \quad (5)$$

$$dT_\tau=+w_r dT_x, \quad (6)$$

the equations for the second family of characteristics:

$$dx=-w_r d\tau, \quad (7)$$

$$dT_\tau=-w_r dT_x, \quad (8)$$

where

$$T_\tau=\frac{\partial T}{\partial\tau},\quad T_x=\frac{\partial T}{\partial x}. \quad (9)$$

Equations (6) and (8) are independent of Eqs. (5) and (7) for the characteristics of their families and can be integrated separately. This yields the two first integrals of the characteristic equations (5)–(8):

$$T_\tau-\int w_r\left(\frac{\partial T}{\partial x}\right)dT_x=\xi, \quad (10)$$

$$T_\tau+\int w_r\left(\frac{\partial T}{\partial x}\right)dT_x=\eta, \quad (11)$$

where ξ and η are constants, which, however, change from characteristic to characteristic.

After finding the first integrals of (10) and (11), the problem reduces to the integration of the system of equations (5) and (7). We will demonstrate that for the integration of this system it is sufficient to solve a system of linear partial-differential equations with variable coefficients.

Let us bring Eqs. (5)–(8) to a new form by an inversion method [5–7]. As the independent variables, for this we will introduce the so-called characteristic variables determined from formulas (10) and (11), and we will regard the old independent variables x and τ as the sought functions of the new variables, i.e., let us consider the transformation

$$x = x(\xi, \eta), \tag{12}$$

$$\tau = \tau(\xi, \eta). \tag{13}$$

Transition to the new variables is possible if the Jacobian of transformations (12) and (13)

$$J = \left(\frac{\partial x}{\partial \xi} \frac{\partial \tau}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial \tau}{\partial \xi} \right) \tag{14}$$

is different from 0.

In this case, instead of the nonlinear equation (4) or its equivalent system of characteristic equations (5)–(8), we obtain two linear equations

$$\frac{\partial x}{\partial \xi} = + w_r(\xi, \eta) \frac{\partial \tau}{\partial \xi}, \tag{15}$$

$$\frac{\partial x}{\partial \eta} = - w_r(\xi, \eta) \frac{\partial \tau}{\partial \eta}, \tag{16}$$

where $w_r(\xi, \eta)$ is the rate of heat propagation expressed in the form of a function of the characteristic variables ξ and η on the basis of formulas (10) and (11).

The system of equations (16) and (11) represents a system of linear partial-differential equations of first order with variable coefficients and is completely equivalent to Eq. (4) if the Jacobian of the transformation is different from 0.

If the Jacobian of the transformation is equal to 0, Eqs. (15) and (16) are not satisfied, since the derivatives

$$\frac{\partial x}{\partial \xi}, \quad \frac{\partial x}{\partial \eta}, \quad \frac{\partial \tau}{\partial \xi}, \quad \frac{\partial \tau}{\partial \eta}$$

may vanish in this case.

However, the case in which the Jacobian of (14) vanishes is represented by the simplest solutions of Eq. (4), since this case describes the so-called simple or shock waves of heat or mass transfer. This case is of independent interest, and we will treat it separately in another paper.

We will present other forms of Eq. (4) or of Eqs. (5)–(8), equivalent to these but reduced to linear equations.

Let us introduce the independent variables u and v in accordance with the formulas

$$u = \frac{\partial T}{\partial \tau}, \quad v = \frac{\partial T}{\partial x}, \tag{17}$$

and regard the old variables x and τ as the sought functions, assuming that the Jacobian of the transformation is not equal to 0:

$$J = \left(\frac{\partial x}{\partial u} \frac{\partial \tau}{\partial v} - \frac{\partial \tau}{\partial u} \frac{\partial x}{\partial v} \right). \tag{18}$$

Using (17) and (18) we bring Eqs. (5)–(7) to equivalent form

$$\frac{\partial x}{\partial u} = w_r(v) \frac{\partial \tau}{\partial u}, \tag{19}$$

$$\frac{\partial x}{\partial v} = - w_r(v) \frac{\partial \tau}{\partial v}, \tag{20}$$

where w_r is a function exclusively of the single variable v. The form of Eqs. (19)–(20) is convenient in the integration of the equations for high-intensity nonsteady-state heat- and mass-transfer.

Equations (15)–(16) or (19)–(20) are easily reduced to linear partial-differential equations of second order with variable coefficients for either one of the functions x or τ.

By differentiation and elimination, instead of (15) and (16) we obtain two second-order equations

$$\frac{\partial^2 x}{\partial \xi \partial \eta} - \frac{1}{2w_r} \frac{dw_r}{d\sigma} \left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta} \right) = 0, \tag{21}$$

$$\frac{\partial^2 \tau}{\partial \xi \partial \eta} + \frac{1}{2w_r} \frac{dw_r}{d\sigma} \left(\frac{\partial \tau}{\partial \xi} + \frac{\partial \tau}{\partial \eta} \right) = 0, \tag{22}$$

where $\sigma = \xi + \eta$ and we assume that $\partial w_r/\partial \xi = \partial w_r/\partial \eta = dw_r/d\sigma$. We can reduce Eqs. (19) and (20) in similar fashion to second-order equations by the same procedure.

Considering Eq. (4) or its equivalent system of equations (5)–(8), (15)–(16), (19)–(20), or (21)–(22), we draw the conclusion that if we assume the rate of propagation for the heat waves to be exclusively a function of the heat flow, the equations of propagation for heat- and mass-transfer will formally coincide with the equations of nonlinear nondissipative and nondispersive electrodynamics [5–7]. This formal analogy can be used for the modeling—by the electrical methods of nonlinear electrodynamics—of the processes of nonlinear high-intensity heat- and mass-transfer.

This formal analogy also indicates new phenomena which will accompany the propagation of the heat- and mass-transfer waves in a nonlinear medium. In particular, we can predict the appearance of heat- and mass-transfer shock waves.

The intensity of the electric field E can be compared to the magnitude of $\partial T/\partial x$; the intensity of the magnetic field H can be compared to the magnitude $\partial T/\partial \tau$ or, conversely, it may depend on the nonlinear electromagnetic medium that is chosen.

Moreover, we can cite the analogy between the processes of intense heat- and mass-transfer and the propagation of elastic-plastic waves [8], gasdynamic waves [9–10], etc.

NOTATION

τ is the time; x is the coordinate; c is the heat capacity; γ is the specific weight of the material; λ is the thermal diffusivity; w_r is the rate of heat (or mass) propagation; τ_r is the time constant.

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